Consensus strategies for cooperative control of vehicle formations

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Abstract: Extensions of a consensus algorithm are introduced for systems modelled by second-order dynamics. Variants of those consensus algorithms are applied to tackle formation control problems by appropriately choosing information states on which consensus is reached. Even in the absence of centralised leadership, the consensus-based formation control strategies can guarantee accurate formation maintenance in the general case of arbitrary (directed) information flow between vehicles as long as certain mild conditions are satisfied. It is shown that many existing leader–follower, behavioural and virtual structure/virtual leader formation control approaches can be unified in the general framework of consensus building. A multiple micro air vehicle formation flying example is shown in simulation to illustrate the strategies.

1 Introduction

Accurate maintenance of a geometric configuration between multiple vehicles moving in formation has been studied extensively in the literature with the hope that through efficient coordination many inexpensive, simple vehicles can achieve better performance than a single monolithic vehicle.

Typical approaches for formation control can be roughly categorised as leader–follower, behavioural and virtual structure/virtual leader approaches, to name a few. In the leader–follower approach [1], some vehicles are designated as leaders and others are designated as followers. The leaders track predefined trajectories, and the followers track transformed versions of the states of their nearest neighbours according to given schemes. In the behavioural approach [2, 3], the control action for each vehicle is defined by a weighted average of the control corresponding to each desired behaviour for the vehicle. In the virtual structure/virtual leader approach [4–8], the entire formation is treated as a single rigid body. The virtual structure can evolve as a whole in a given direction with some given orientation and maintain a rigid geometric relationship among multiple vehicles.

As an inherently distributed strategy to multi-vehicle coordination, information consensus has received significant attention in the control community recently. The basic idea for information consensus is that each vehicle updates its information state on the basis of the information states of its local neighbours in such a way that the final information state of each vehicle converges to a common value. Consensus algorithms and their convergence analyses have recently been studied in various works [9–14]. Those algorithms take the form of first-order dynamics. Extensions to second-order dynamics under undirected information flow are discussed in the works of Tanner *et al.* [15, 16] and Olfati-Saber and Murray [17].

Through appropriately choosing information states on which consensus is reached, consensus algorithms can be applied to tackle formation control problems. We propose consensus-based formation control strategies that guarantee accurate formation keeping with only local neighbour-to-neighbour information exchange. In particular, we show that many existing leader–follower, behavioural, and virtual structure/virtual leader approaches in the literature can be considered special cases of consensus-based formation control strategies. In the consensus building framework, the leader–follower approach [1] corresponds to the case that consensus is reached on each sum of a position vector and an inter-vehicle separation vector and the information flow is itself a (directed) spanning tree. The behavioural approach [3] corresponds to the case that consensus is reached on each deviation vector between the actual vehicle location and the desired vehicle location and the information flow forms a bidirectional ring topology. The decentralised virtual structure approach [18] corresponds to the case that consensus is reached on each instantiation of the virtual structure states and the information flow forms a bidirectional ring topology. As a comparison, in the more general consensus-based formation control framework, formation keeping is guaranteed as long as a subset of the (possibly directed) information flow topology forms a (directed) spanning tree. As a result, group robustness can be introduced to the leader–follower approach by allowing information flow from the followers to the leaders. For the behavioural and decentralised virtual structure approach, more flexility in intervehicle information exchange is introduced in the sense that (i) there is no need to identify two adjacent neighbours, (ii) sensors with limited fields of views can be employed and (iii) packet loss for some information exchange links may be accounted for.

The main contribution of this paper is to introduce extensions of a consensus algorithm for systems modelled by second-order dynamics, which extends the first-order consensus algorithms in the literature, and apply variants of those algorithms to formation control problems by appropriately choosing the information states on which consensus is

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reached. We show that many existing leader–follower, behavioural and virtual structure/virtual leader formation control approaches can be unified in the general framework of consensus building. The benefit of this unification is that the consensus building framework only requires local neighbour-to-neighbour information exchange and naturally takes into account arbitrary information flow between the vehicles by allowing information to flow from any vehicle to any other vehicle to introduce feedback/coupling (and therefore increase redundancy and robustness for the whole team) without complexifying the control law design and convergence/stability analysis. Taking into account the case that sensors may have limited fields of views, we consider the general case that information flow is unidirectional or directed. As a result, bidirectional or directed information flow is a special case of the unidirectional one.

2 Background and preliminaries

It is natural to model information exchange between vehicles by directed/undirected graphs. A digraph (directed graph) consists of a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite non-empty set of nodes and $\mathcal{E} \in \mathcal{N}^2$ is a set of ordered pairs of nodes, called edges. As a comparison, the pairs of nodes in an undirected graph are unordered. If there is a directed edge from node v_i to node v_j, then v_i is defined as the parent node and v_j is defined as the child node. A directed path is a sequence of ordered edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots$, where $v_{i_j} \in \mathcal{N}$, in a digraph. An undirected path in an undirected graph is defined accordingly. A digraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is called connected if there is a path between any distinct pair of nodes. A directed tree is a digraph, where every node has exactly one parent except for one node, called the root, which has no parent, but has a directed path to every other node. A (directed) spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph. We say that a graph has (or contains) a (directed) spanning tree if there exists a (directed) spanning tree being a subset of the graph. Note that the condition that a digraph has a (directed) spanning tree is equivalent to the case that there exists at least one node having a directed path to all the other nodes. In the case of undirected graphs, having an undirected spanning tree is equivalent to being connected. However, in the case of directed graphs, having a directed spanning tree is a weaker condition than being strongly connected.

Let $\mathbf{1}$ and $\mathbf{0}$ denote the $n \times 1$ column vector of all ones and all zeros, respectively. Let $\boldsymbol{I}_n$ denote the $n \times n$ identity matrix and $\mathbf{0}_{m\times n}$ denote the $m \times n$ matrix with all zero entries. Let $M_n(\mathbb{R})$ represent the set of all $n \times n$ real matrices. Given a matrix $\boldsymbol{A} = [a_{ij}] \in M_n(\mathbb{R})$, the digraph of $\boldsymbol{A}$, denoted by $\Gamma(\boldsymbol{A})$, is the digraph on n nodes v_i, $i \in \{1, 2, \ldots, n\}$, such that there is a directed edge in $\Gamma(\boldsymbol{A})$ from v_j to v_i if and only if $a_{ij} \neq 0$ [19].

The adjacency matrix $\boldsymbol{A} = [a_{ij}] \in M_n(\mathbb{R})$ of a weighted digraph is defined as $a_{ii} = 0$ and $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ where $i \neq j$. The Laplacian matrix of the weighted digraph is defined as $\boldsymbol{L} = [\ell_{ij}] \in M_n(\mathbb{R})$, where $\ell_{ii} = \sum_{j\neq i} a_{ij}$ and $\ell_{ij} = -a_{ij}$ where $i \neq j$. For an undirected graph, the Laplacian matrix is symmetric positive semi-definite. This property does not hold for a digraph Laplacian matrix.

In the case of an undirected information exchange graph, the graph Laplacian has a simple zero eigenvalue and all the other eigenvalues are positive if and only if the graph is connected [20]. In the case of a directed information exchange graph, the digraph Laplacian has a simple zero eigenvalue and all the other eigenvalues have positive real parts if and only if the digraph has a (directed) spanning tree [21]. In both cases, $\mathbf{1}$ is the eigenvector of the graph (digraph) Laplacian associated with eigenvalue zero.

3 Fundamental consensus algorithm and its extensions

3.1 Basic results

In this section, we focus on consensus algorithms for systems modelled by second-order dynamics.

Let $\boldsymbol{\xi}_i \in \mathbb{R}^m$ and $\boldsymbol{\zeta}_i \in \mathbb{R}^m$ be the information states of the ith vehicle. For example, $\boldsymbol{\xi}_i$ may take the role of position, altitude or heading angle and $\boldsymbol{\zeta}_i$ may take the role of velocity, climb rate or angular velocity of the ith vehicle.

For information states with second-order dynamics, a fundamental second-order consensus algorithm is proposed by Ren and Atkins [22] as follows

$$\dot{\boldsymbol{\xi}}_i = \boldsymbol{\zeta}_i$$
$$\dot{\boldsymbol{\zeta}}_i = -\sum_{j=1}^{n} g_{ij}k_{ij}[(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j) + \gamma(\boldsymbol{\zeta}_i - \boldsymbol{\zeta}_j)], \quad i \in \{1, \ldots, n\} \tag{1}$$

where $k_{ij} > 0$, $\gamma > 0$, $g_{ii} \triangleq 0$ and g_{ij} is 1, if information flows from vehicle j to vehicle i and 0 otherwise.

For consensus algorithm (1), consensus is said to be reached asymptotically among multiple vehicles if $\|\boldsymbol{\xi}_i(t) - \boldsymbol{\xi}_j(t)\| \to 0$ and $\|\boldsymbol{\zeta}_i(t) - \boldsymbol{\zeta}_j(t)\| \to 0$, $\forall i \neq j$, as $t \to \infty$ for any $\boldsymbol{\xi}_i(0)$ and $\boldsymbol{\zeta}_i(0)$.

If it is desirable to guarantee that $\boldsymbol{\xi}_i \to \boldsymbol{\xi}_j$ and $\boldsymbol{\zeta}_i \to \boldsymbol{\zeta}_j \to \boldsymbol{\zeta}^*(t)$, where $\boldsymbol{\zeta}^*(t) \in \mathbb{R}^m$ is a reference for $\boldsymbol{\zeta}_i$, we propose an extension of consensus algorithm (1) as follows

$$\dot{\boldsymbol{\xi}}_i = \boldsymbol{\zeta}_i \tag{2}$$
$$\dot{\boldsymbol{\zeta}}_i = \dot{\boldsymbol{\zeta}}^* - \alpha(\boldsymbol{\zeta}_i - \boldsymbol{\zeta}^*) - \sum_{j=1}^{n} g_{ij}k_{ij}[(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j) + \gamma(\boldsymbol{\zeta}_i - \boldsymbol{\zeta}_j)]$$

where $\alpha > 0$.

If it is desirable to guarantee that $\boldsymbol{\xi}_i \to \boldsymbol{\xi}_j \to \boldsymbol{\xi}^*(t)$ and $\boldsymbol{\zeta}_i \to \boldsymbol{\zeta}_j \to \boldsymbol{\zeta}^*(t)$, where $\boldsymbol{\xi}^*(t) \in \mathbb{R}^m$ and $\boldsymbol{\zeta}^*(t) \in \mathbb{R}^m$ are references for $\boldsymbol{\xi}_i$ and $\boldsymbol{\zeta}_i$, respectively, and satisfy $\dot{\boldsymbol{\xi}}^*(t) = \boldsymbol{\zeta}^*(t)$, we propose another extension of consensus algorithm (1) as follows

$$\dot{\boldsymbol{\xi}}_i = \boldsymbol{\zeta}_i \tag{3}$$
$$\dot{\boldsymbol{\zeta}}_i = \dot{\boldsymbol{\zeta}}^* - \beta[(\boldsymbol{\xi}_i - \boldsymbol{\xi}^*) + \gamma(\boldsymbol{\zeta}_i - \boldsymbol{\zeta}^*)] - \sum_{j=1}^{n} g_{ij}k_{ij}[(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j) + \gamma(\boldsymbol{\zeta}_i - \boldsymbol{\zeta}_j)]$$

where $\beta > 0$.

In the following, we assume $m = 1$ for simplicity. However, all the results hereafter remain valid for $m > 1$. Before moving on, we need the following two lemmas.

Lemma 3.1: Let $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ represent the real and imaginary parts of a number, respectively. Also let

$$\rho_{\pm} = \frac{\gamma\mu - \alpha \pm \sqrt{(\gamma\mu - \alpha)^2 + 4\mu}}{2}$$

where ρ, $\mu \in \mathbb{C}$. If $\alpha \geq 0$, $\mathrm{Re}(\mu) < 0$, and

$$\gamma > \sqrt{\frac{2}{|\mu| \cos((\pi/2) - \tan^{-1}(-\mathrm{Re}(\mu)/\mathrm{Im}(\mu)))}}$$

then $\mathrm{Re}(\rho_{\pm}) < 0$.

Proof: Refer to the work of Ren and Atkins [22]. □

Lemma 3.2: Let $\boldsymbol{L} \in M_n(\mathbb{R})$ be a digraph Laplacian matrix. Let $\boldsymbol{p}$ be a left eigenvector of $-\boldsymbol{L}$ associated with eigenvalue 0 and $\boldsymbol{p}^{\mathrm{T}}\mathbf{1} = 1$. Also let

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{I}_n \\ -\boldsymbol{L} & -\alpha \boldsymbol{I}_n - \gamma \boldsymbol{L} \end{bmatrix}$$

where $\alpha > 0$ and $\gamma > 0$. Then

$$\lim_{t\to\infty} \mathrm{e}^{\boldsymbol{\Sigma} t} \to \begin{bmatrix} \mathbf{1}\boldsymbol{p}^{\mathrm{T}} & \frac{1}{\alpha}\mathbf{1}\boldsymbol{p}^{\mathrm{T}} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}$$

if and only if $\boldsymbol{\Sigma}$ has a simple zero eigenvalue and all the other eigenvalues have negative real parts.

Proof: Refer to the work of Ren and Atkins [22]. □

Corollary 3.1: Let $\boldsymbol{L} = [\ell_{ij}] \in M_n(\mathbb{R})$ denote a digraph Laplacian matrix, where $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$ and $\ell_{ij} = -g_{ij}k_{ij}$, $\forall i \neq j$. Let $\boldsymbol{p}$ be a left eigenvector of $-L$ associated with eigenvalue 0 and $\boldsymbol{p}^{\mathrm{T}}\mathbf{1} = 1$. Also let $\xi = [\xi_1, \ldots, \xi_n]^{\mathrm{T}}$ and $\zeta = [\zeta_1, \ldots, \zeta_n]^{\mathrm{T}}$. With consensus algorithm (2), $\xi_i(t) \to \boldsymbol{p}^{\mathrm{T}} \xi(0) + \frac{1}{\alpha}\ \boldsymbol{p}^{\mathrm{T}}(\zeta(0) - 1\zeta^*(0)) + \int_0^t \zeta^*(\tau)\ \mathrm{d}\tau$ and $\zeta_i(t) \to \zeta_j(t) \to \zeta^*(t)$, $\forall i$, asymptotically as $t \to \infty$ if and only if matrix $\boldsymbol{\Sigma}$ has a simple zero eigenvalue and all the other eigenvalues have negative real parts.

Proof (Sufficiency): Let $\tilde{\xi}_i = \xi_i - \xi^*$, where $\xi^* = \int_0^t \zeta^*(\tau)\mathrm{d}\tau$, and $\tilde{\zeta}_i = \zeta_i - \zeta^*$. From (2), we know that

$$\begin{aligned} \dot{\xi}_i - \dot{\xi}^* &= \zeta_i - \zeta^* \\ \dot{\zeta}_i &= -\dot{\zeta}^* - \alpha(\zeta_i - \zeta^*) - \sum_{j=1}^{n} g_{ij}k_{ij}\{(\xi_i - \xi^*) \\ &\quad - (\xi_j - \xi^*) + \gamma[(\zeta_i - \zeta^*) - (\zeta_j - \zeta^*)]\} \end{aligned}$$

which implies that

$$\begin{aligned} \dot{\tilde{\xi}}_i &= \tilde{\zeta}_i \\ \dot{\tilde{\zeta}}_i &= -\alpha\tilde{\zeta}_i - \sum_{j=1}^{n} g_{ij}k_{ij}[(\tilde{\xi}_i - \tilde{\xi}_j) + \gamma(\tilde{\zeta}_i - \tilde{\zeta}_j)] \end{aligned} \tag{4}$$

Equation (4) can be written in matrix form as

$$\begin{bmatrix} \dot{\tilde{\xi}} \\ \dot{\tilde{\zeta}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0}_{n\times n} & \boldsymbol{I}_n \\ -\boldsymbol{L} & -\alpha\boldsymbol{I}_n - \gamma L \end{bmatrix}}_{\Sigma} \begin{bmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{bmatrix} \tag{5}$$

where $\tilde{\xi} = [\tilde{\xi}_1, \ldots, \tilde{\xi}_n]^{\mathrm{T}}$, $\tilde{\zeta} = [\tilde{\zeta}_1, \ldots, \tilde{\zeta}_n]^{\mathrm{T}}$ and $\boldsymbol{L} = [\ell_{ij}]$ denotes the digraph Laplacian matrix with $\ell_{ii} = \sum_{j\neq i} g_{ij}k_{ij}$ and $\ell_{ij} = -g_{ij}k_{ij}$, $\forall i \neq j$.

If $\boldsymbol{\Sigma}$ has a simple zero eigenvalue and all the other eigenvalues have negative real parts, we know from Lemma 3.2 that

$$\begin{bmatrix} \tilde{\xi}(t) \\ \tilde{\zeta}(t) \end{bmatrix} \to \begin{bmatrix} \mathbf{1}\boldsymbol{p}^{\mathrm{T}} & \frac{1}{\alpha}\mathbf{1}\boldsymbol{p}^{\mathrm{T}} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix} \begin{bmatrix} \tilde{\xi}(0) \\ \tilde{\zeta}(0) \end{bmatrix}$$

which implies that $\tilde{\boldsymbol{\xi}}(t) \to \mathbf{1}\boldsymbol{p}^{\mathrm{T}}\tilde{\xi}(0) + \frac{1}{\alpha}\mathbf{1}\boldsymbol{p}^{\mathrm{T}}\tilde{\zeta}(0)$ and $\tilde{\zeta}(t) \to \mathbf{0}$. This in turn gives the sufficient part.

(Necessity) If $\boldsymbol{\xi}_i(t) \to \boldsymbol{p}^{\mathrm{T}}\xi(0) + \frac{1}{\alpha}\boldsymbol{p}^{\mathrm{T}}(\boldsymbol{\zeta}(0) - \mathbf{1}\zeta^*(0)) + \int_0^t \zeta^*(\tau)\,\mathrm{d}\tau$ and $\zeta_i(t) \to \zeta_j(t) \to \zeta^*(t)$ asymptotically as $t \to \infty$, we know that $\tilde{\xi}(t) \to \mathbf{1}\boldsymbol{p}^{\mathrm{T}}\tilde{\xi}(0) + \frac{1}{\alpha}\mathbf{1}\boldsymbol{p}^{\mathrm{T}}\ \tilde{\boldsymbol{\zeta}}(0)$ and $\tilde{\zeta}(t) \to \mathbf{0}$, which implies that

$$\lim_{t\to\infty} \mathrm{e}^{\boldsymbol{\Sigma} t} \to \begin{bmatrix} \mathbf{1}\boldsymbol{p}^{\mathrm{T}} & \frac{1}{\alpha}\mathbf{1}\boldsymbol{p}^{\mathrm{T}} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}$$

The necessary part then comes from Lemma 3.2. □

We also have the following two theorems for consensus algorithms (2) and (3).

Theorem 3.2: Let L and $\boldsymbol{p}$ be defined in Corollary 3.1. Also let μ_i be the eigenvalues of $-L$. Consensus algorithm (2) guarantees that $\xi_i(t) \to \xi_j(t) \to \boldsymbol{p}^{\mathrm{T}}\xi(0) + \frac{1}{\alpha}\boldsymbol{p}^{\mathrm{T}}(\zeta(0) - \mathbf{1}\zeta^*(0)) + \int_0^t \zeta^*(\tau)\,\mathrm{d}\tau$ and $\zeta_i(t) \to \zeta_j(t) \to \zeta^*(t)$, $\forall i$, asymptotically if the information exchange topology has a (directed) spanning tree and

$$\gamma > \max_{\{i|\mathrm{Re}(\mu_i)<0\}} \sqrt{\frac{2}{|\mu_i| \cos((\pi/2) - \tan^{-1}(-\mathrm{Re}(\mu_i)/\mathrm{Im}(\mu_i)))}} \tag{6}$$

where $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ represent the real and imaginary parts of a number, respectively.

Proof: Given (5), we can solve the equation $\det(\lambda\boldsymbol{I}_{2n} - \boldsymbol{\Sigma}) = 0$ to find the eigenvalues of $\boldsymbol{\Sigma}$. Note that

$$\begin{aligned} \det(\lambda\boldsymbol{I}_{2n} - \boldsymbol{\Sigma}) &= \det(\lambda^2\boldsymbol{I}_n + \gamma\lambda\boldsymbol{L} + \alpha\lambda\boldsymbol{I}_n + \boldsymbol{L}) \\ &= \det((\lambda^2 + \alpha\lambda)\boldsymbol{I}_n + (1 + \gamma\lambda)\boldsymbol{L}) \end{aligned} \tag{7}$$

Also note that

$$\det(\lambda\boldsymbol{I}_n + \boldsymbol{L}) = \prod_{i=1}^{n}(\lambda - \mu_i) \tag{8}$$

where μ_i is the ith eigenvalue of $-\boldsymbol{L}$.

By comparing (7) and (8), we see that the roots of (7) can be obtained by solving $\lambda^2 + \alpha\lambda = \mu_i(1 + \gamma\lambda)$. Therefore it is straightforward to see that the eigenvalues of $\boldsymbol{\Sigma}$ are given by

$$\rho_{i\pm} = \frac{\gamma\mu_i - \alpha \pm \sqrt{(\gamma\mu_i - \alpha)^2 + 4\mu_i}}{2}$$

where $\rho_{i\pm}$ are called eigenvalues of $\boldsymbol{\Sigma}$ that are associated with μ_i.

If the information exchange topology has a (directed) spanning tree, we know that $-\boldsymbol{L}$ has a simple zero eigenvalue and all the other eigenvalues have negative real parts [21]. Without loss of generality, we let $\mu_1 = 0$ and $\mathrm{Re}(\mu_i) < 0$, $i = 2, \ldots, n$. Then we can obtain $\rho_{1+} = 0$ and $\rho_{1-} = -\alpha$. If Inequality (6) is true, we know that $\mathrm{Re}(\rho_{i\pm}) < 0$, $i = 2, \ldots, n$, from Lemma 3.1. Therefore we see that consensus can be achieved asymptotically from Corollary 3.1. □

Theorem 3.3: Let $\boldsymbol{L}$ and $\boldsymbol{p}$ be defined in Corollary 3.1. Also let μ_i be the eigenvalues of $-\boldsymbol{L}$. Consensus algorithm (3) guarantees that $\xi_i(t) \to \xi^*(t)$ and $\zeta_i(t) \to \zeta^*(t)$, $\forall i$, asymptotically, if

$$\gamma > \max_{i=1,\ldots,n} \sqrt{\frac{2}{|\nu_i| \cos((\pi/2) - \tan^{-1}(-\mathrm{Re}(\nu_i)/\mathrm{Im}(\nu_i)))}} \quad (9)$$

where $\nu_i = -\beta + \mu_i$.

Proof: By following the proof of Corollary 3.1, we rewrite (3) as

$$\begin{bmatrix} \dot{\tilde{\xi}} \\ \dot{\tilde{\zeta}} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{0}_{n\times n} & \boldsymbol{I}_n \\ -(\beta \boldsymbol{I} + \boldsymbol{L}) & -\gamma(\beta \boldsymbol{I} + \boldsymbol{L}) \end{bmatrix}}_{\Upsilon} \begin{bmatrix} \tilde{\xi} \\ \tilde{\zeta} \end{bmatrix}$$

where $\tilde{\xi} = \xi - \mathbf{1}\xi^*$ and $\tilde{\zeta} = \zeta - \mathbf{1}\zeta^*$. Note that eigenvalues of $-(\beta \boldsymbol{I} + L)$ are given by ν_i, where $\nu_i = -\beta + \mu_i$ and $\mathrm{Re}(\nu_i) < 0$, $\forall i$, for any information exchange topology. Following the proof of Theorem 3.2, eigenvalues of Υ are given by $\rho_{i\pm} = (\gamma\nu_i \pm \sqrt{(\gamma^2\nu_i^2 + 4\nu_i)})/2$. If Inequality (9) is true, we know that all eigenvalues of Υ have negative real parts from Lemma 3.1. Therefore we see that $\tilde{\xi} \to \mathbf{0}$ and $\tilde{\zeta} \to \mathbf{0}$, which in turn proves the theorem. □

Note that unlike Theorem 3.2, the information exchange topology does not affect the convergence result in Theorem 3.3 as long as the scaling factor γ is sufficiently large. As a result, even in the worst case that there is no information exchange between the vehicles (i.e. $\boldsymbol{L} = 0_{n\times n}$), we can still guarantee that $\xi_i \to \xi^*$ and $\zeta_i \to \zeta^*$, $\forall i$, as long as Inequality (9) is valid. However, better transient performance is achieved when the information exchange topology has a (directed) spanning tree due to the coupling between the vehicles.

3.2 Discussions

Note that algorithms (2) and (3) represent the fundamental forms of second-order consensus algorithms. These algorithms can be extended to achieve different convergence results as shown in the next section.

Also note that Theorems 3.2 and 3.3 focus on a time-invariant information exchange topology. Theorem 3.2 relies on the assumption that the information exchange topology has a (directed) spanning tree and the scaling factor γ is above a certain lower bound, and Theorem 3.3 relies on the assumption that γ is above another lower bound. In the case of a time-invariant information exchange topology, the consensus algorithms are exponentially stable, which implies that the consensus algorithms are robust to information exchange noise.

In the case of dynamically changing information exchange topologies, the following condition motivated by Ren and Atkins [22] is required to guarantee the convergence result of the consensus algorithms.

Let $t_0, t_1, \ldots$ be the times when the information exchange topology switches. Also let τ be the dwell time such that $t_{i+1} - t_i \geq \tau$, $\forall i = 0, 1, \ldots$. If the information exchange topology has a (directed) spanning tree for each $t \in [t_i, t_{i+1})$, the condition for γ in (6) is satisfied for each t_i, and the dwell time τ is sufficiently large, then consensus algorithm (2) achieves consensus exponentially and is robust to information exchange noise under switching (directed) information exchange topologies. If the condition for γ in (9) is satisfied for each t_i, and the dwell time τ is sufficiently large, then consensus algorithm (3) achieves consensus exponentially and is robust to information exchange noise under switching (directed) information exchange topologies.

Note that the above condition is mild and takes into account the fact that the information links between the vehicles may be broken or established randomly in practice.

4 Consensus strategies for formation control

In this section, we apply consensus algorithms (2) and (3) to design formation control strategies. The consensus algorithms and their variants will be used in the context of leader–follower, behavioural and virtual structure/virtual leader approaches. We will show that many existing results using these three approaches can be unified in the general framework of consensus building.

We assume that the dynamics of each vehicle are

$$\dot{\boldsymbol{r}}_i = \boldsymbol{v}_i, \quad \dot{\boldsymbol{v}}_i = \boldsymbol{u}_i$$

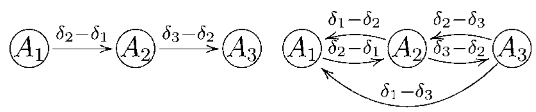
where $\boldsymbol{r}_i \in \mathbb{R}^m$ and $\boldsymbol{v}_i \in \mathbb{R}^m$ represent the position and velocity of vehicle i, and $\boldsymbol{u}_i \in \mathbb{R}^m$ is the control input.

In the first case, a variant of consensus algorithm (2) is applied to guarantee that $\boldsymbol{r}_i - \mathbf{r_j} \to \boldsymbol{\Delta}_{ij}$ and $\boldsymbol{v}_i \to \boldsymbol{v}_j \to \boldsymbol{v}^d(t)$, where $\boldsymbol{v}^d(t)$ specifies the nominal formation velocity and $\boldsymbol{\Delta}_{ij}$ denotes the desired separation between vehicle i and vehicle j. Letting $\boldsymbol{\delta}_i \in \mathbb{R}^m$ be constants, the control input is designed as

$$\boldsymbol{u}_i = \dot{\boldsymbol{v}}^d - \alpha(\boldsymbol{v}_i - \boldsymbol{v}^d) - \sum_{j=1}^{n} g_{ij}k_{ij}[(\boldsymbol{r}_i - \delta_i) - (\boldsymbol{r}_j - \delta_j) + \gamma(\boldsymbol{v}_i - \boldsymbol{v}_j)] \quad (10)$$

Note that $\boldsymbol{r}_i - \boldsymbol{\delta}_i$ and $\boldsymbol{v}_i$ satisfy consensus algorithm (2) with $\boldsymbol{r}_i - \boldsymbol{\delta}_i$, $\boldsymbol{v}_i$ and $\boldsymbol{v}^d$ playing the role of ξ_i, ζ_i and ζ^*, respectively. Then we know that $\boldsymbol{r}_i - \delta_i \to \boldsymbol{r}_j - \boldsymbol{\delta}_j$ and $\boldsymbol{v}_i \to \boldsymbol{v}_j \to \boldsymbol{v}^d(t)$, if the conditions in Theorem 3.2 are satisfied. That is, $\boldsymbol{r}_i - \boldsymbol{r}_j \to \boldsymbol{\delta}_i - \boldsymbol{\delta}_j$ and $\boldsymbol{v}_i \to \boldsymbol{v}_j \to \boldsymbol{v}^d(t)$. Therefore $\boldsymbol{\delta}_i$ can be chosen such that desired separations between vehicles are guaranteed. Note that (10) can be extended to account for the case that $\boldsymbol{\delta}_i$ is time-varying. In particular, the leader–follower approach in the work of Wang and Hadaegh [1] corresponds to the case that information only flows from leaders to followers. In a leader–follower scenario shown in Fig. 1*a*, if the information flow from a leader to a follower breaks, the team fails. Although it is possible to introduce information links from followers to leaders in the leader–follower approach [1], it is not straightforward how the stability analysis of the whole team will be affected. As a comparison, the consensus building framework provides a way to introduce information flow from followers to leaders as shown in Fig. 1*b* without affecting the stability analysis. The coupling between the vehicles serves as a form of team feedback, which improves team robustness without affecting formation maintenance accuracy.

In the second case, a variant of consensus algorithm (2) is used to achieve formation maintenance via the behavioural approach [2, 3]. Let $\{\boldsymbol{r}_i^* | i = 1, \ldots, n\}$ specify the desired (possibly time-varying) formation shape. Fig. 2 shows



a *b*

Fig. 1 *Formation keeping with desired separations*

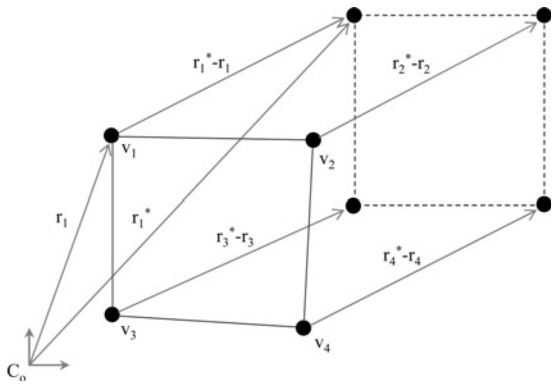


Fig. 2 *Consensus reached on deviation vectors*

a scenario where multiple vehicles reach consensus on $\boldsymbol{r}_i^* - \boldsymbol{r}_i$.

Note that if $\boldsymbol{r}_i^* - \boldsymbol{r}_i$ reaches a common value, then the desired formation shape is guaranteed to be preserved. The control input in this case is designed as

$$\begin{aligned}\boldsymbol{u}_i = {} & \ddot{\boldsymbol{r}}_i^* + \dot{\boldsymbol{v}}^d - \alpha(\boldsymbol{v}_i - \dot{\boldsymbol{r}}_i^* - \boldsymbol{v}^d) \\ & - \sum_{j=1}^n g_{ij}k_{ij}[(\boldsymbol{r}_i - \boldsymbol{r}_i^*) - (\boldsymbol{r}_j - \boldsymbol{r}_j^*)] \\ & - \sum_{j=1}^n g_{ij}k_{ij}\gamma[(\boldsymbol{v}_i - \dot{\boldsymbol{r}}_i^*) - (\boldsymbol{v}_j - \dot{\boldsymbol{r}}_j^*)] \qquad (11)\end{aligned}$$

where $\boldsymbol{v}^d$ specifies the nominal formation velocity, the third term is a damping term, the last two terms are used to guarantee that the desired formation shape between vehicles is preserved (formation keeping behaviour). Note that $\boldsymbol{r}_i - \boldsymbol{r}_i^*$ and $\boldsymbol{v}_i - \dot{\boldsymbol{r}}_i^*$ satisfy consensus algorithm (2) with $\boldsymbol{r}_i - \boldsymbol{r}_i^*$, $\boldsymbol{v}_i - \dot{\boldsymbol{r}}_i^*$ and $\boldsymbol{v}^d$ playing the role of ξ_i, ζ_i, and ζ^*, respectively. Then we know that $\boldsymbol{r}_i - \boldsymbol{r}_i^* \to \boldsymbol{r}_j - \boldsymbol{r}_j^*$ and $\boldsymbol{v}_i - \dot{\boldsymbol{r}}_i^* \to \boldsymbol{v}_j - \dot{\boldsymbol{r}}_j^* \to \boldsymbol{v}^d$ if the conditions in Theorem 3.2 are satisfied.

In particular, if it is desirable that each vehicle reaches its desired location eventually while preserving the desired formation shape during the transition, we apply a variant of consensus algorithm (3) to design the control input as follows

$$\begin{aligned}\boldsymbol{u}_i = {} & \ddot{\boldsymbol{r}}_i^* - \beta(\boldsymbol{r}_i - \boldsymbol{r}_i^*) - \gamma\beta(\boldsymbol{v}_i - \dot{\boldsymbol{r}}_i^*) \\ & - \sum_{j=1}^n g_{ij}k_{ij}[(\boldsymbol{r}_i - \boldsymbol{r}_i^*) - (\boldsymbol{r}_j - \boldsymbol{r}_j^*)] \\ & - \sum_{j=1}^n g_{ij}k_{ij}\gamma[(\boldsymbol{v}_i - \dot{\boldsymbol{r}}_i^*) - (\boldsymbol{v}_j - \dot{\boldsymbol{r}}_j^*)] \qquad (12)\end{aligned}$$

where the second and third terms are used to guarantee that each vehicle arrives at its destination (goal seeking behaviour), and the last two terms are used to guarantee that the desired formation shape between vehicles is preserved (formation keeping behaviour).

Letting $\boldsymbol{r} = [\boldsymbol{r}_1^{\mathrm{T}}, \ldots, \boldsymbol{r}_n^{\mathrm{T}}]^{\mathrm{T}}$, $\boldsymbol{v} = [\boldsymbol{v}_1^{\mathrm{T}}, \ldots, \boldsymbol{v}_n^{\mathrm{T}}]^{\mathrm{T}}$, $\boldsymbol{r}^* = [\boldsymbol{r}_1^{*\mathrm{T}}, \ldots, \boldsymbol{r}_n^{*\mathrm{T}}]^{\mathrm{T}}$, and $\boldsymbol{v}^* = [\dot{\boldsymbol{r}}_1^{*\mathrm{T}}, \ldots, \dot{\boldsymbol{r}}_n^{*\mathrm{T}}]^{\mathrm{T}}$, we obtain

$$\begin{bmatrix}\dot{\tilde{\boldsymbol{r}}} \\ \dot{\tilde{\boldsymbol{v}}}\end{bmatrix} = \left(\underbrace{\begin{bmatrix}\boldsymbol{0}_{n\times n} & \boldsymbol{I}_n \\ -(\beta\boldsymbol{I}_n + \boldsymbol{L}) & -\gamma(\beta\boldsymbol{I}_n + \boldsymbol{L})\end{bmatrix}}_{\Delta} \otimes \boldsymbol{I}_m\right)\begin{bmatrix}\tilde{\boldsymbol{r}} \\ \tilde{\boldsymbol{v}}\end{bmatrix}$$

where $\tilde{\boldsymbol{r}} = \boldsymbol{r}^* - \boldsymbol{r}$, $\tilde{\boldsymbol{v}} = \boldsymbol{v}^* - \boldsymbol{v}$ and $\otimes$ denotes the Kronecker product. We see that $\tilde{\boldsymbol{r}} \to \boldsymbol{0}_{mn\times 1}$ and $\tilde{\boldsymbol{v}} \to \boldsymbol{0}_{mn\times 1}$, that is $\boldsymbol{r}_i \to \boldsymbol{r}_i^*$ and $\boldsymbol{v}_i \to \dot{\boldsymbol{r}}_i^*$, if the conditions in Theorem 3.3 are satisfied. That is, even if the information exchange topology does not have a (directed) spanning tree (even the worse case that $\boldsymbol{L} = \boldsymbol{0}_{n\times n}$), all vehicles will reach their (possibly time-varying) destinations eventually. However, the desired formation shape is not guaranteed to be preserved during the transition. The last two terms in (12) with the graph of $\boldsymbol{L}$ having a (directed) spanning tree are important to guarantee formation keeping during the transition.

In the third case, a variant of consensus algorithm (2) is used to decentralise the virtual structure/virtual leader approach. In a centralised scheme, the state of the virtual coordinate frame or virtual leader is implemented at a central location (e.g. a ground station or a leader vehicle) and broadcast to every vehicle in the team. The state of the virtual coordinate frame or virtual leader then serves as a reference for each vehicle to derive local control laws. Although this implementation may be feasible in the case that a robust central location exists and high bandwidth communication is available, issues such as a single point of failure or stringent intervehicle communication constraints will significantly degrade the overall system performance. One remedy to these limitations is to instantiate a local copy of the state of the virtual coordinate frame on each vehicle. If each vehicle implements the same cooperation algorithm, we expect that the decentralised scheme achieves the same cooperation as the centralised one. However, because dynamically changing local situation awareness (e.g. disturbances, noise, environmental changes etc.) for each vehicle, there exist discrepancies among each instantiation of the state of the virtual coordinate frame. In this case, our approach is to apply a consensus algorithm to drive each instantiation of the state of the virtual coordinate frame to converge to a sufficiently common value as well as design local control strategies such that the actual states of each vehicle track its desired ones.

In the context of virtual structures/virtual leaders, we continuously update the virtual structure/virtual leader instantiation on each vehicle according to a consensus strategy of the form

$$\begin{aligned}\dot{\boldsymbol{r}}_{\mathrm{F}i} &= \boldsymbol{v}_{\mathrm{F}i} \\ \dot{\boldsymbol{v}}_{\mathrm{F}i} &= \dot{\boldsymbol{v}}_{\mathrm{F}}^d - \alpha(\boldsymbol{v}_{\mathrm{F}i} - \boldsymbol{v}_{\mathrm{F}}^d) + \kappa_i(\boldsymbol{x}_i, \boldsymbol{x}_i^d) \\ & \quad - \sum_{j=1}^n g_{ij}k_{ij}[(\boldsymbol{r}_{\mathrm{F}i} - \boldsymbol{r}_{\mathrm{F}j}) + \gamma(\boldsymbol{v}_{\mathrm{F}i} - \boldsymbol{v}_{\mathrm{F}j})]\end{aligned}$$

where $[\boldsymbol{r}_{\mathrm{F}i}^{\mathrm{T}}, \boldsymbol{v}_{\mathrm{F}i}^{\mathrm{T}}]^{\mathrm{T}}$ is the ith instantiation of the virtual structure/virtual leader state (i.e. formation centre, formation velocity), $\boldsymbol{v}_{\mathrm{F}}^d \in \mathbb{R}^m$ specifies the nominal formation velocity, $\boldsymbol{x}_i = [\boldsymbol{r}_i^{\mathrm{T}}, \boldsymbol{v}_i^{\mathrm{T}}]^{\mathrm{T}}$ is the local state of vehicle i, $\boldsymbol{x}_i^d = \boldsymbol{\psi}_i(t, \boldsymbol{x}_i, \boldsymbol{r}_{\mathrm{F}i}, \boldsymbol{v}_{\mathrm{F}i})$ is the desired local state of vehicle i, and $\boldsymbol{\kappa}_i(\cdot,\cdot)$ denotes the group feedback term introduced from the ith vehicle to the ith instantiation of the virtual structure/virtual leader state. The introduction of $\boldsymbol{\kappa}_i$ is to adjust the evolution speed of the ith instantiation of the virtual structure/virtual leader state according to formation performance (e.g. formation accuracy). As a simple example, $\boldsymbol{\kappa}_i(\cdot,\cdot)$ may be a function of $e_i = ||\boldsymbol{x}_i - \boldsymbol{x}_i^d||$. If e_i is large (small), which implies that vehicle i cannot (can) accurately track its desired state in the presence (absence) of saturation constraints or disturbance, we can make κ_i negative (positive) so as to slow down (speed up) the evolution speed of the virtual structure/virtual leader in the case that the ith vehicle is behind its desired location. As a result, the ith vehicle can catch up its desired location in the case that e_i is large or move at a higher speed in the case that e_i is small.

The local control of the ith vehicle is designed as $\boldsymbol{u}_i = \chi_i(t, \boldsymbol{x}_i, \boldsymbol{x}_i^d)$ such that $\boldsymbol{x}_i \to \boldsymbol{x}_i^d$. Note that $\boldsymbol{u}_i$ only depends on its own local state information and virtual structure/virtual leader instantiation.

Given the conditions in Theorem 3.2, it can be shown that if $||\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j||$, $\forall i \neq j$, is bounded, so are $||\boldsymbol{r}_{\mathrm{F}i} - \boldsymbol{r}_{\mathrm{F}j}||$ and $||\boldsymbol{v}_{\mathrm{F}i} - \boldsymbol{v}_{\mathrm{F}j}||$. In particular, if $||\boldsymbol{\kappa}_i||$, $\forall i$, is bounded, so are $||\boldsymbol{r}_{\mathrm{F}i} - \boldsymbol{r}_{\mathrm{F}j}||$ and $||\boldsymbol{v}_{\mathrm{F}i} - \boldsymbol{v}_{\mathrm{F}j}||$.

As an alternative, a variant of consensus algorithm (2) can be used to derive local control laws for each vehicle directly such that they agree on a common (time-varying) formation centre. Let $\boldsymbol{r}_j$ be the jth vehicle's position. Let $\boldsymbol{r}_{0j}$ be the jth vehicle's understanding of the formation centre. Also let $\boldsymbol{r}_{j\mathrm{F}}$ be the desired deviation of the jth vehicle from its understanding of the formation centre. Note that $\boldsymbol{r}_j = \boldsymbol{r}_{0j} + \boldsymbol{r}_{j\mathrm{F}}$. Fig. 3 shows a scenario where multiple vehicles reach consensus on a (possibly time-varying) formation centre. If $\boldsymbol{r}_{0j}$ reaches a common value, denoted as $\boldsymbol{r}_0$, then the desired formation shape is preserved as $\boldsymbol{r}_j \to \boldsymbol{r}_0 + \boldsymbol{r}_{j\mathrm{F}}$. In this case, the control input is designed as

$$\begin{aligned}\boldsymbol{u}_i = {} & \ddot{\boldsymbol{r}}_{i\mathrm{F}} + \dot{\boldsymbol{v}}_{\mathrm{F}}^d - \alpha(\boldsymbol{v}_i - \dot{\boldsymbol{r}}_{i\mathrm{F}} - \boldsymbol{v}_{\mathrm{F}}^d) \\ & - \sum_{j=1}^{n} g_{ij}k_{ij}[(\boldsymbol{r}_i - \boldsymbol{r}_{i\mathrm{F}}) - (\boldsymbol{r}_j - \boldsymbol{r}_{j\mathrm{F}})] \\ & - \sum_{j=1}^{n} g_{ij}k_{ij}\gamma[(\boldsymbol{v}_i - \dot{\boldsymbol{r}}_{i\mathrm{F}}) - (\boldsymbol{v}_j - \dot{\boldsymbol{r}}_{j\mathrm{F}})] \end{aligned} \qquad (13)$$

Note that $\boldsymbol{r}_i - \boldsymbol{r}_{i\mathrm{F}}$ and $\boldsymbol{v}_i - \dot{\boldsymbol{r}}_{i\mathrm{F}}$ satisfy consensus algorithm (2) with $\boldsymbol{r}_i - \boldsymbol{r}_{i\mathrm{F}}$, $\boldsymbol{v}_i - \dot{\boldsymbol{r}}_{i\mathrm{F}}$, and $\boldsymbol{v}_{\mathrm{F}}^d$ playing the role of ξ_i, ζ_i and ζ^*, respectively. Then we know that $\boldsymbol{r}_i - \boldsymbol{r}_{i\mathrm{F}} \to \boldsymbol{r}_j - \boldsymbol{r}_{j\mathrm{F}}$ and $\boldsymbol{v}_i - \boldsymbol{r}_{i\mathrm{F}} \to \boldsymbol{v}_j - \dot{\boldsymbol{r}}_{j\mathrm{F}} \to \boldsymbol{v}_{\mathrm{F}}^d$ if the conditions in Theorem 3.2 are satisfied. That is, $\boldsymbol{r}_{0i} \to \boldsymbol{r}_{0j}$ and $\dot{\boldsymbol{r}}_{0i} \to \dot{\boldsymbol{r}}_{0j} \to \boldsymbol{v}_{\mathrm{F}}^d$.

5 Application to formation flying of multiple micro air vehicles

In this section, we apply consensus strategies to coordinate the flight of multiple rotary-wing micro air vehicles (MAVs) to form a sensor web with a time-varying desired geometric configuration. Owing to space limitations, we only show results using control law (13).

Let (x_i, y_i, h_i), ψ_i, v_i, r_i and v_{hi} denote the three-dimensional inertial position, heading angle, forward velocity, heading rate and vertical velocity of the ith rotary-wing MAV, respectively. With the MAV equipped with efficient low-level controllers, the simplified equations of motion are given by

$$\begin{aligned} \dot{x}_i &= v_i \cos(\psi_i) \\ \dot{y}_i &= v_i \sin(\psi_i) \\ \dot{\psi}_i &= r_i \\ \dot{v}_i &= \frac{1}{\alpha_{vi}}(v_i^c - v_i) \\ \dot{r}_i &= \frac{1}{\alpha_{ri}}(r_i^c - r_i) \\ \dot{h}_i &= v_{hi} \\ \dot{v}_{hi} &= \frac{1}{\alpha_{v_{hi}}}(v_{hi}^c - v_{hi}) \end{aligned} \qquad (14)$$

where v_i^c, r_i^c and v_{hi}^c are the commanded forward velocity, heading rate and vertical velocity to the low-level controllers, and α_* are positive constants [23]. Assuming that effective altitude-hold controllers guarantee that the MAVs fly at the same constant altitude, we will focus on the design of velocity and heading rate control commands in the following.

To avoid the non-holonomic constraint introduced by (14), we define

$$\begin{bmatrix} x_{fi} \\ y_{fi} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} d_i \cos(\psi_i) \\ d_i \sin(\psi_i) \end{bmatrix}$$

Note that if (x_i, y_i) represents MAV i's lateral CG position in inertial coordinates, (x_{fi}, y_{fi}) represents the inertial position of a point f_i located a distance d_i along the x body axis of the ith MAV, presuming zero pitch angle. In the following, we will focus on the coordination of (x_{fi}, y_{fi}) instead of (x_i, y_i) to simplify design of the coordination algorithms.

Motivated by Lawton *et al.* [3], if we let

$$\begin{bmatrix} v_i^c \\ r_i^c \end{bmatrix} = \begin{bmatrix} v_i \\ r_i \end{bmatrix} + \begin{bmatrix} \alpha_{vi} & 0 \\ 0 & \alpha_{ri} \end{bmatrix} \begin{bmatrix} \cos(\psi_i) & -d_i \sin(\psi_i) \\ \sin(\psi_i) & d_i \cos(\psi_i) \end{bmatrix}^{-1} \times \begin{bmatrix} \mu_{xi} + v_i r_i \sin(\psi_i) + d_i r_i^2 \cos(\psi_i) \\ \mu_{yi} - v_i r_i \cos(\psi_i) + d_i r_i^2 \sin(\psi_i) \end{bmatrix}$$

we obtain the following equations of motion

$$\begin{aligned} \dot{\boldsymbol{r}}_{fi} &= \boldsymbol{v}_{fi} \\ \dot{\boldsymbol{v}}_{fi} &= \boldsymbol{\mu}_{fi} \end{aligned} \qquad (15)$$

where $\boldsymbol{r}_{fi} = [x_{fi}, y_{fi}]^{\mathrm{T}}$ and $\boldsymbol{\mu}_{fi} = [\mu_{xi}, \mu_{yi}]^{\mathrm{T}}$. Note that the transformation between (μ_{xi}, μ_{yi}) and (v_i^c, r_i^c) are invertible. Also note that the stability analysis of the internal dynamics of the closed-loop system in (14) and (15) is identical to that of the internal dynamics of a feedback linearised non-holonomic mobile robot dynamic model described in the work of Lawton *et al.* [3]. It has been shown [3] that the zero dynamics of the latter is stable.

We apply control law (13) to design $\boldsymbol{\mu}_{fi}$ such that a team of four MAVs fly with a pre-defined formation velocity given by $\boldsymbol{v}_{\mathrm{F}}^d(t)$ and the team preserves a time-varying square geometric configuration during the flight.

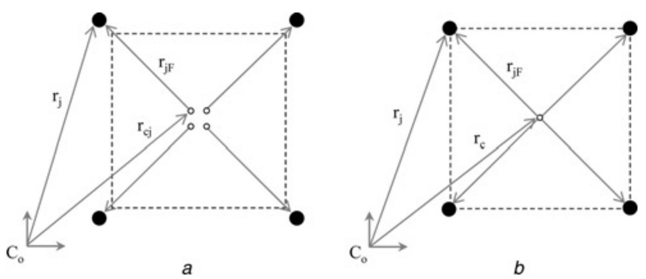


Fig. 3 *Consensus reached on a (possibly time-varying) formation centre*

The parameter values used in the simulation are given by Table 1, where

$$\lambda(t) = \begin{cases} \dfrac{t}{100} + \dfrac{1}{2}, & t < 50\,\text{s} \\ 1, & t \geq 50\,\text{s} \end{cases}$$

$$\phi(t) = \begin{cases} 0, & t < 50\,\text{s} \\ \dfrac{t-50}{15}, & 50 \leq t < 50 + \dfrac{15\pi}{2}\,\text{s} \\ \dfrac{\pi}{2}, & t \geq 50 + \dfrac{15\pi}{2}\,\text{s} \end{cases}$$

and

$$\boldsymbol{R}(\phi(t)) = \begin{bmatrix} \cos(\phi(t)) & \sin(\phi(t)) \\ -\sin(\phi(t)) & \cos(\phi(t)) \end{bmatrix}$$

Note that the size of the desired square geometric configuration between the four MAVs will be expanded at $t \in [0,\ 50)$ seconds as shown by the definitions of $\lambda(t)$ and $\boldsymbol{r}_{i\mathrm{F}}(t)$, $i = 1, \ldots, 4$.

The information exchange topologies between the four MAVs are given in Fig. 4, where a directed edge from the ith MAV to the jth MAV means that the jth MAV can receive information from the ith MAV. Taking into account measurements from sensors with limited fields of views or random communication data loss, we assume a unidirectional information flow topology. Note that Fig. 4*a* has a (directed) spanning tree, whereas Fig. 4*b* does not have a (directed) spanning tree. (Multiple leaders exist in this case.)

We will consider three cases. Table 2 gives control parameters for each case.

Figs. 5–7 show the trajectories of the four MAVs in cases 1, 2 and 3, respectively, where squares represent the actual starting positions of each MAV, respectively ($t = 0$ s), circles represent the actual ending positions of each MAV ($t = 100$ s) and triangles represent the actual positions of each MAV at $t = \{25;\ 50;\ 75\}$ s. Note that the team preserves the desired time-varying square formation and flies with a nominal formation velocity given by $\boldsymbol{v}_{\mathrm{F}}^{d}$ in case 1. However, the desired time-varying square formation

Table 1: Parameter values used in simulation

Parameter	Value
α_{vi}	1
α_{ri}	1
k_{ij}	1
v_i^c	$\in[-3,\ 3]$ m/s
r_i^c	$\in[-1,\ 1]$ rad/s
$\boldsymbol{v}_{\mathrm{F}}^{d}$	$2 * [\sin(\phi),\ \cos(\phi)]^{\mathrm{T}}$
$\boldsymbol{r}_{1\mathrm{F}}$	$\lambda(t)\boldsymbol{R}(\phi(t))\ [10,\ 10]^{\mathrm{T}}$
$\boldsymbol{r}_{2\mathrm{F}}$	$\lambda(t)\boldsymbol{R}(\phi(t))\ [-10,\ 10]^{\mathrm{T}}$
$\boldsymbol{r}_{3\mathrm{F}}$	$\lambda(t)\boldsymbol{R}(\phi(t))\ [-10,\ -10]^{\mathrm{T}}$
$\boldsymbol{r}_{4\mathrm{F}}$	$\lambda(t)\boldsymbol{R}(\phi(t))\ [-10,\ 10]^{\mathrm{T}}$

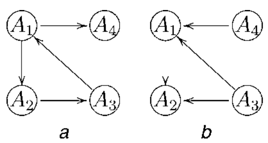


Fig. 4 *Information exchange topologies between the four MAVs*

Table 2: Control parameters for different cases

Case 1: Interaction graph: Fig. 4*a* $\alpha = 1$, $\gamma = 1$	
Case 2: Interaction graph: Fig. 4*b* $\alpha = 1$, $\gamma = 1$	
Case 3: Interaction graph: Fig. 4*a* $\alpha = 0.5$, $\gamma = 0.05$	

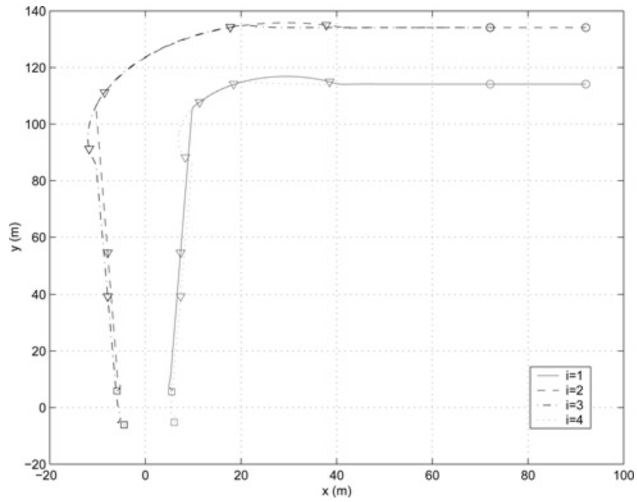


Fig. 5 *Trajectories of the four MAVs in case 1*

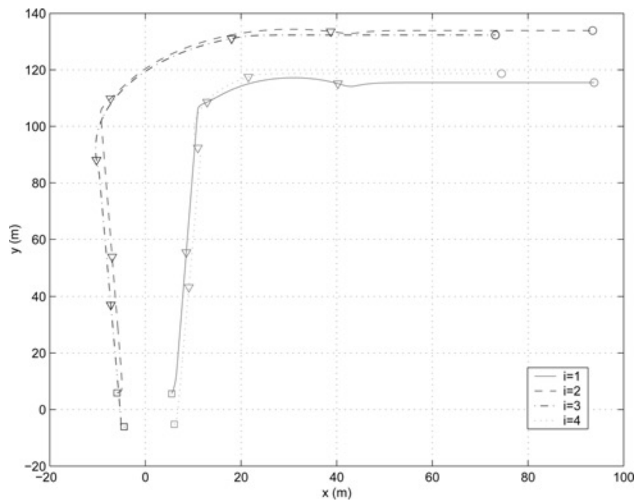


Fig. 6 *Trajectories of the four MAVs in case 2*

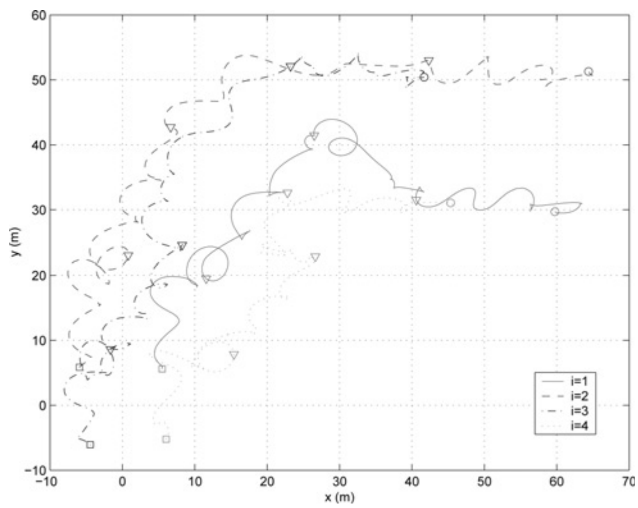


Fig. 7 *Trajectories of the four MAVs in case 3*

is not preserved in either case 2 or case 3 due to the lack of a (directed) spanning tree in Fig. 4*b* in case 2 and small γ and α in case 3.

6 Conclusion

We have introduced extensions of a consensus algorithm for systems modelled by second-order dynamics and applied variants of those algorithms to formation control problems by appropriately choosing the information states on which consensus is reached. We have shown that many existing leader–follower, behavioural, and virtual structure/virtual leader formation control approaches can be thought of as special cases of the consensus-based strategies. An application to multiple MAV formation flying has been given to demonstrate the effectiveness of our strategies.

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8 References

1 Wang, P.K.C., and Hadaegh, F.Y.: 'Coordination and control of multiple microspacecraft moving in formation', *J. Astronaut. Sci.*, 1996, **44**, (3), pp. 315–355
2 Balch, T., and Arkin, R.C.: 'Behavior-based formation control for multirobot teams', *IEEE Trans. Robot. Autom.*, 1998, **14**, (6), pp. 926–939
3 Lawton, J.R., Beard, R.W., and Young, B.: 'A decentralized approach to formation maneuvers', *IEEE Trans. Robot. Autom.*, 2003, **19**, (6), pp. 933–941
4 Lewis, M.A., and Tan, K.-H.: 'High precision formation control of mobile robots using virtual structures', *Auton. Robot.*, 1997, **4**, pp. 387–403
5 Beard, R.W., Lawton, J.R., and Hadaegh, F.Y.: 'A coordination architecture for spacecraft formation control', *IEEE Trans. Control Syst. Technol.*, 2001, **9**, (6), pp. 777–790
6 Kang, W., Xi, N., and Sparks, A.: 'Formation control of autonomous agents in 3D workspace'. Proc. IEEE Int. Conf. on Robotics and Automation, San Francisco, CA, April 2000, pp. 1755–1760
7 Leonard, N.E., and Fiorelli, E.: 'Virtual leaders, artificial potentials and coordinated control of groups'. Proc. IEEE Conf. on Decision and Control, Orlando, Florida, December 2001, pp. 2968–2973
8 Egerstedt, M., Hu, X., and Stotsky, A.: 'Control of mobile platforms using a virtual vehicle approach', *IEEE Trans. Autom. Control*, 2001, **46**, (11), pp. 1777–1782
9 Jadbabaie, A., Lin, J., and Morse, A.S.: 'Coordination of groups of mobile autonomous agents using nearest neighbor rules', *IEEE Trans. Autom. Control*, 2003, **48**, (6), pp. 988–1001
10 Lin, Z., Broucke, M., and Francis, B.: 'Local control strategies for groups of mobile autonomous agents', *IEEE Trans. Autom. Control*, 2004, **49**, (4), pp. 622–629
11 Olfati-Saber, R., and Murray, R.M.: 'Consensus problems in networks of agents with switching topology and time-delays', *IEEE Trans. Autom. Control*, 2004, **49**, (9), pp. 1520–1533
12 Moreau, L.: 'Stability of multi-agent systems with time-dependent communication links', *IEEE Trans. Autom. Control*, 2005, **50**, (2), pp. 169–182
13 Ren, W., and Beard, R.W.: 'Consensus seeking in multiagent systems under dynamically changing interaction topologies', *IEEE Trans. Autom. Control*, 2005, **50**, (5), pp. 655–661
14 Moore, K., and Lucarelli, D.: 'Forced and constrained consensus among cooperating agents'. IEEE Int. Conf. on Networking, Sensing and Control, Tucson, AZ, 2005
15 Tanner, H.G., Jadbabaie, A., and Pappas, G.J.: 'Stable flocking of mobile agents, Part i: fixed topology'. Proc. IEEE Conf. on Decision and Control, Maui, Hawaii, December 2003, pp. 2010–2015
16 Tanner, H.G., Jadbabaie, A., and Pappas, G.J.: 'Stable flocking of mobile agents, Part ii: dynamic topology'. Proc. IEEE Conf. on Decision and Control, Maui, Hawaii, December 2003, pp. 2016–2021
17 Olfati-Saber, R., and Murray, R.M.: 'Flocking with obstacle avoidance: cooperation with limited communication in mobile networks'. Proc. IEEE Conf. on Decision and Control, Maui, Hawaii, December 2003, pp. 2022–2028
18 Ren, W., and Beard, R.W.: 'Decentralized scheme for spacecraft formation flying via the virtual structure approach', *AIAA J. Guid. Control Dyn.*, 2004, **27**, (1), pp. 73–82
19 Horn, R.A., and Johnson, C.R.: 'Matrix analysis' (Cambridge University Press, 1985)
20 Chung, F.R.K.: 'Spectral graph theory', 'Regional Conference Series in Mathematics' (American Mathematical Society, 1997)
21 Ren, W., Beard, R.W., and McLain, T.W.: 'Coordination variables and consensus building in multiple vehicle systems' in Kumar, V., Leonard, N.E., and Morse, A.S. (Eds.): 'Cooperative control: a post-workshop volume 2003 block island workshop on cooperative control' (Springer-Verlag Series: Lecture Notes in Control and Information Sciences, 2005), vol. 309, pp. 171–188
22 Ren, W., and Atkins, E.M.: 'Distributed multi-vehicle coordinated control via local information exchange', *Int. J. Robust Nonlinear Control*, 2006, in press
23 Gavrilets, V., Mettler, B., and Feron, E.: 'Human-inspired control logic for automated maneuvering of miniature helicopter', *AIAA J. Guid., Control Dyn.*, 2004, **27**, (5), pp. 752–759